

Signed degree sets in signed bipartite graphs

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AMS Classification: 05C

Abstract

A signed bipartite graph $G(U, V)$ is a bipartite graph in which each edge is assigned a positive or a negative sign. The signed degree of a vertex x in $G(U, V)$ is the number of positive edges incident with x less the number of negative edges incident with x . The set S of distinct signed degrees of the vertices of $G(U, V)$ is called its signed degree set. In this paper, we prove that every set of integers is the signed degree set of some connected signed bipartite graph.

1. Introduction

A signed graph is a graph in which each edge is assigned a positive or a negative sign. The concept of signed graphs is given by Harary [3]. Let G be a signed graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$. The signed degree of a vertex v_i in G is denoted by $sdeg(v_i)$ (or simply by d_i) and is defined as $d_i = d_i^+ - d_i^-$, where $1 \leq i \leq n$ and d_i^+ (d_i^-) is the number of positive (negative) edges incident with v_i . A signed degree sequence $\sigma = [d_1, d_2, \dots, d_n]$ of a signed graph G is formed by listing the vertex signed degrees in non-increasing order. An integral sequence is s-graphical if it is the signed degree sequence of a signed graph. Also, a non-zero sequence $\sigma = [d_1, d_2, \dots, d_n]$ is a standard sequence if σ is non-increasing, $\sum_{i=1}^n d_i$ is even, $d_1 > 0$, each $|d_i| < n$, and $|d_1| \geq |d_n|$.

The following result [1], gives a necessary and sufficient condition for an integral sequence to be s-graphical, which is similar to Hakimi's result for degree sequences in graphs [2].

Theorem 1.1. A standard integral sequence $\sigma = [d_1, d_2, \dots, d_n]$ is s-graphical if and only if

$\sigma' = [d_2 - 1, d_3 - 1, \dots, d_{d_1+s+1} - 1, d_{d_1+s+2}, \dots, d_{n-s}, d_{n-s+1} + 1, \dots, d_n + 1,]$
is s-graphical for some s, $0 \leq s \leq \frac{n-1-d_1}{2}$.

The next characterization for signed degree sequences in signed graphs is due to Yan et al. [8].

Theorem 1.2. A standard integral sequence $\sigma = [d_1, d_2, \dots, d_n]$ is s-graphical

if and only if

$\sigma'_m = [d_2 - 1, d_3 - 1, \dots, d_{d_1+m+1} - 1, d_{d_1+m+2}, \dots, d_{n-m}, d_{n-m+1} + 1, \dots, d_n + 1,]$
is s-graphical, where m is the maximum non-negative integer such that $d_{d_1+m+1} > d_{n-m+1}$

The set of distinct signed degrees of the vertices of a signed graph is called its signed degree set.

In [4], Kapoor et al. proved that every set of positive integers is the degree set of some connected graph and determined the smallest order for such a graph. Pirzada et al. [7] proved that every set of positive (negative) integers is the signed degree set of some connected signed graph and determined the smallest possible order for such a signed graph.

A graph G is called bipartite if its vertex set can be partitioned into two nonempty disjoint subsets U and V such that each edge in G joins a vertex in U with a vertex in V, and is denoted by $G(U, V)$. Let $G(U, V)$ be a bipartite graph with $U = \{u_1, u_2, \dots, u_p\}$ and $V = \{v_1, v_2, \dots, v_q\}$. Then, degree of u_i (v_j) is the number of edges of $G(U, V)$ incident with u_i (v_j) and is denoted by $\deg u_i$ or simply by d_i ($\deg v_j$ or simply by e_j). Then, the sequences $[d_1, d_2, \dots, d_p]$ and $[e_1, e_2, \dots, e_q]$ are called the degree sequences of $G(U, V)$. The set of distinct degrees of the vertices of a bipartite graph $G(U, V)$ is called its degree set.

The criteria for bipartite graphical sequences is given by Gale-Ryser theorem

$$\sum_{i=1}^p d_i = \sum_{j=1}^q e_j, \quad \sum_{i=1}^k d_i \leq \sum_{j=1}^q \min\{k, e_j\}, \text{ for } 1 \leq k \leq p.$$

Pirzada et al. [6] proved that every set of non-negative integers is a degree set of some bipartite graph.

A signed bipartite graph is a bipartite graph in which each edge is assigned a positive or a negative sign. Let $G(U, V)$ be a signed bipartite graph with $U = \{u_1, u_2, \dots, u_p\}$ and $V = \{v_1, v_2, \dots, v_q\}$. Then,

$$\text{signed degree of } u_i \text{ is } s\deg(u_i) = d_i = d_i^+ - d_i^-,$$

where $1 \leq i \leq p$ and d_i^+ (d_i^-) is the number of positive (negative) edges incident with u_i , and

signed degree of v_j is $\text{sdeg}(v_j) = e_j = e_j^+ - e_j^-$,

where $1 \leq j \leq q$ and e_j^+ (e_j^-) is the number of positive (negative) edges incident with v_j .

Clearly, $|d_i| \leq q$ and $|e_j| \leq p$. Then, the sequences $\alpha = [d_1, d_2, \dots, d_p]$ and $\beta = [e_1, e_2, \dots, e_q]$ are called the signed degree sequences of the signed bipartite graph $G(U, V)$. Two sequences $\alpha = [d_1, d_2, \dots, d_p]$ and $\beta = [e_1, e_2, \dots, e_q]$ are said to be standard sequences if

Two sequences of integers $\alpha = [d_1, d_2, \dots, d_p]$ and $\beta = [e_1, e_2, \dots, e_q]$ are said to be standard sequences if

(i) α is non-zero, (ii) α is non-increasing and $|d_1| \geq |d_p|$, for we may always replace α and β by $-\alpha$ and $-\beta$ if necessary, (iii) $\sum_{i=1}^p d_i = \sum_{j=1}^q e_j$, (iv) $d_1 > 0$, (v) each $|d_i| \leq q$, each $|e_j| \leq p$, and each $|e_j| \leq |d_1|$.

or (i) β is non-zero, (ii) β is non-increasing and $|e_1| \geq |e_q|$, for we may always replace α and β by $-\alpha$ and $-\beta$ if necessary, (iii) $\sum_{i=1}^p d_i = \sum_{j=1}^q e_j$, (iv) $e_1 > 0$, (v) each $|d_i| \leq q$, each $|e_j| \leq p$, and each $|d_i| \leq |e_1|$.

Also, a signed bipartite graph $G(U, V)$ is said to be connected if each vertex $u \in U$ is connected to every vertex $v \in V$.

The next result, due to Pirzada and Naikoo [5], is a necessary and sufficient condition for a pair of integral sequences to be the signed degree sequences of some signed bipartite graph.

Theorem 1.3. Let $\alpha = [d_1, d_2, \dots, d_p]$ and $\beta = [e_1, e_2, \dots, e_q]$ be standard sequences. Then, α and β are the signed degree sequences of a signed bipartite graph if and only if there exist integers r and s with $d_1 = r - s$ and $0 \leq s \leq \frac{q - d_1}{2}$ such that α' and β' are the signed degree sequences of a signed bipartite graph, where α' is obtained from α by deleting d_1 and β' is obtained from β by reducing r greatest entries of β by 1 each and adding s least entries of β by 1 each.

For any two sets X and Y , we denote by $X \oplus Y$ to mean that each vertex of X is joined to every vertex of Y by a positive edge.

2. Main Results

The set S of distinct signed degrees of the vertices of a signed bipartite graph $G(U, V)$ is called its signed degree set.

The following result shows that every set of positive integers is a signed degree set of some connected signed bipartite graph.

Theorem 2.1. Let d_1, d_2, \dots, d_n be positive integers. Then, there exists a connected signed bipartite graph with signed degree set $S = \{d_1, \sum_{i=1}^2 d_i, \dots, \sum_{i=1}^n d_i\}$.

Proof. If $n = 1$, then a signed bipartite graph $G(U, V)$ with $|U| = |V| = d_1$ and $U \oplus V$ has signed degree set $S = \{d_1\}$.

For $n \geq 2$, construct a signed bipartite graph $G(U, V)$ as follows.

Let $U = X_1 \cup X_2 \cup X_2' \cup \dots \cup X_n \cup X_n'$,

$V = Y_1 \cup Y_2 \cup Y_2' \cup \dots \cup Y_n \cup Y_n'$,

with $X_i \cap X_j = \emptyset$, $X_i \cap X_j' = \emptyset$, $X_i' \cap X_j' = \emptyset$, $Y_i \cap Y_j = \emptyset$, $Y_i \cap Y_j' = \emptyset$, $Y_i' \cap Y_j' = \emptyset$ ($i \neq j$),

$|X_i| = |Y_i| = d_i$ for all i , $1 \leq i \leq n$, $|X_i'| = |Y_i'| = d_1 + d_2 + \dots + d_{i-1}$ for all i , $2 \leq i \leq n$. Let (i) $X_i \oplus Y_j$ whenever $i \geq j$, (ii) $X_i' \oplus Y_i$ for all i , $2 \leq i \leq n$, and (iii) $X_i' \oplus Y_i'$ for all i , $2 \leq i \leq n$. Then, the signed degrees of the vertices of $G(U, V)$ are as follows.

For $1 \leq i \leq n$

$$\text{sdeg}(x_i) = \sum_{j=1}^i |Y_j| = \sum_{j=1}^i d_j = d_1 + d_2 + \dots + d_i, \text{ for all } x_i \in X_i,$$

for $2 \leq i \leq n$

$$\text{sdeg}(x_i') = |Y_i| + |Y_i'| = d_i + d_1 + d_2 + \dots + d_{i-1}$$

$$= d_1 + d_2 + \dots + d_i, \text{ for all } x_i' \in X_i', \text{ for } 1 \leq i \leq n,$$

$$\text{sdeg}(y_i) = \sum_{j=i}^n |X_j| + |X_i'| = \sum_{j=i}^n d_j + d_1 + d_2 + \dots + d_{i-1}$$

$$= d_i + d_{i+1} + \dots + d_n + d_1 + d_2 + \dots + d_{i-1}$$

$$= d_1 + d_2 + \dots + d_n, \text{ for all } y_i \in Y_i,$$

and for $2 \leq i \leq n$

$$\text{sdeg}(y_i') = |X_i'| = d_1 + d_2 + \dots + d_{i-1}, \text{ for all } y_i' \in Y_i'.$$

Therefore, signed degree set of $G(U, V)$ is $S = \left\{ d_1, \sum_{i=1}^2 d_i, \dots, \sum_{i=1}^n d_i \right\}$.

Clearly, by construction, all the signed bipartite graphs are connected. Hence, the result follows. \blacksquare

By interchanging positive edges with negative edges in Theorem 2.1, we obtain the following result.

Corollary 2.1. Every set of negative integers is a signed degree set of some connected signed bipartite graph.

Finally, we have the following result.

Theorem 2.2. Every set of integers is a signed degree set of some connected signed bipartite graph.

Proof. Let S be a set of integers. Then, we have the following five cases.

(i) S is a set of positive (negative) integers. Then, the result follows by Theorem 2.1 (Corollary 2.1).

(ii) $S = \{0\}$. Then, a signed bipartite graph $G(U, V)$ with $|U| = |V| = 2$ in which u_1v_1 , u_2v_2 are positive edges and u_1v_2 , u_2v_1 are negative edges, where $u_1, u_2 \in U$, $v_1, v_2 \in V$, has signed degree set S .

(iii) S is a set of non-negative (non-positive) integers. Let $S = S_1 \cup \{0\}$, where S_1 is a set of positive (negative) integers. Then, by Theorem 2.1

(Corollary 2.1), there is a connected signed bipartite graph $G_1(U_1, V_1)$ with signed degree set S_1 . Construct a new signed bipartite graph $G(U, V)$ as follows.

Let $U = U_1 \cup \{x_1\} \cup \{x_2\}$,

$V = V_1 \cup \{y_1\} \cup \{y_2\}$,

with $U_1 \cap \{x_i\} = \phi$, $\{x_1\} \cap \{x_2\} = \phi$, $V_1 \cap \{y_i\} = \phi$, $\{y_1\} \cap \{y_2\} = \phi$.

Let u_1y_1 , x_1v_1 , x_2y_2 be positive edges and u_1y_2 , x_1y_1 , x_2v_1 be negative edges, where $u_1 \in U_1$ and $v_1 \in V_1$. Then, $G(U, V)$ has degree set S . We note that addition of such edges do not effect the signed degrees of the vertices of $G_1(U_1, V_1)$, and the vertices x_1 , x_2 , y_1 , y_2 have signed degrees zero each.

(iv) S is a set of non-zero integers. Let $S = S_1 \cup S_2$, where S_1 and S_2 are sets of positive and negative integers respectively. Then, by Theorem 2.1 and Corollary 2.1, there are connected signed bipartite graphs $G_1(U_1, V_1)$ and $G_2(U_2, V_2)$ with signed degree sets S_1 and S_2 respectively. Let $G'_1(U'_1, V'_1)$ and $G'_2(U'_2, V'_2)$ be the copies of $G_1(U_1, V_1)$ and $G_2(U_2, V_2)$ with signed degree sets S_1 and S_2 respectively. Construct a new signed bipartite graph $G(U, V)$ as follows.

Let $U = U_1 \cup U'_1 \cup U_2 \cup U'_2$,

$V = V_1 \cup V'_1 \cup V_2 \cup V'_2$,

with $U_i \cap U'_j = \phi$, $U_1 \cap U_2 = \phi$, $U'_1 \cap U'_2 = \phi$, $V_i \cap V'_j = \phi$, $V_1 \cap V_2 = \phi$, $V'_1 \cap V'_2 = \phi$. Let $u_1v'_2$, u'_1v_2 be positive edges and u_1v_2 , $u'_1v'_2$ be negative edges, where $u_i \in U_i$, $v_i \in V_i$, $u'_i \in U'_i$ and $v'_i \in V'_i$. Then, $G(U, V)$ has signed degree set S . We note that addition of such edges do not effect the signed degrees of the vertices of $G_1(U_1, V_1)$, $G'_1(U'_1, V'_1)$, $G_2(U_2, V_2)$ and $G'_2(U'_2, V'_2)$.

(v) S is a set of all integers. Let $S = S_1 \cup S_2 \cup \{0\}$, where S_1 and S_2 are sets of positive and negative integers respectively. Then, by Theorem 2.1 and Corollary 2.1, there exist connected signed bipartite graphs $G_1(U_1, V_1)$ and $G_2(U_2, V_2)$ with signed degree sets S_1 and S_2 respectively. Construct a new signed bipartite graph $G(U, V)$ as follows.

Let $U = U_1 \cup U_2 \cup \{x\}$,

$V = V_1 \cup V_2 \cup \{y\}$,

with $U_1 \cap U_2 = \phi$, $U_i \cap \{x\} = \phi$, $V_1 \cap V_2 = \phi$, $V_i \cap \{y\} = \phi$. Let u_1v_2 , u_2y , xv_1 be positive edges and u_1y , u_2v_1 , xv_2 be negative edges, where $u_i \in U_i$ and $v_i \in V_i$. Then, $G(U, V)$ has signed degree set S . We note that addition of such edges do not effect the signed degrees of the vertices of $G_1(U_1, V_1)$ and $G_2(U_2, V_2)$, and the vertices x and y have signed degrees zero each.

Clearly, by construction, all the signed bipartite graphs are connected. This proves the result. \blacksquare

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